

Imprimitive Distance-Regular Graphs and Projective Planes

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Communicated by W. T. Tutte

Received March 8, 1973

We investigate a class of (imprimitive) covering graphs Γ of complete bipartite graphs $K_{k,k}$ and show that they are in one-to-one correspondence with triples (\mathbf{P}, l, P) , where \mathbf{P} is a projective plane of order k and (l, P) is a distinguished flag of \mathbf{P} . If Γ is distance-transitive, then $\mathbf{P} - l$ is a self-dual rank three translation plane and may be coordinatised by a semifield.

Many of the central theorems in the theory of permutation groups are proved under the assumption that a certain group acts primitively. Such assumptions are mostly unacceptable in the study of graphs, since we are left with the problems, first of identifying (by means of some graph theoretical properties) those graphs which admit such a primitive group of automorphisms, and second of characterizing the imprimitive cases in terms of the primitive ones. In the theory of permutation groups the first problem is empty by definition, and the second is handled by considering the action induced by an arbitrary given permutation group on some set of maximal blocks. In the theory of graphs the latter problem may not be so easy, since the graphical structure (that is, the edge-set) may not be compatible with the equivalence relation induced by the set of blocks. We apply to an arbitrary distance-regular graph a combinatorial definition of imprimitivity, for which the block size cannot exceed the valency; we show that in the extreme case when the block size is equal to the valency, k say, either $k = 3$ or there exists a projective plane of order k .

1. INTRODUCTION

Γ denotes throughout a regular, undirected, simple, connected graph of diameter d and valency k , with vertex-set $V\Gamma$. If $\alpha \in V\Gamma$, then $\Gamma_i(\alpha)$ denotes the set of vertices of Γ at distance i from α , $0 \leq i \leq d$; we write $\Gamma_1(\alpha) =$

$\Gamma(\alpha)$. Γ is *distance-regular* if for each i , $0 \leq i \leq d$, and any $\beta \in \Gamma_i(\alpha)$ the following numbers depend only on i .

$$c_i = |\Gamma(\beta) \cap \Gamma_{i-1}(\alpha)|; \quad a_i = |\Gamma(\beta) \cap \Gamma_i(\alpha)|; \\ b_i = |\Gamma(\beta) \cap \Gamma_{i+1}(\alpha)|.$$

c_0 and b_d are not defined, whereas $b_0 = k$, $c_d = 1$. For the rest of this paper we shall work with some distance-regular graph Γ ; in particular the following integers are well defined.

$$k_i = |\Gamma_i(\alpha)| = (b_0 \cdot b_1 \cdot \dots \cdot b_{i-1}) / (c_1 \cdot c_2 \cdot \dots \cdot c_i)$$

has *intersection array*

$$i(\Gamma) = \{b_0, b_1, \dots, b_{d-1}; c_1, c_2, \dots, c_d\}.$$

A graph Γ is *G-distance-transitive*, for some group G of automorphisms of Γ , if for each i , $0 \leq i \leq d$, G acts transitively on pairs (α, β) , where $\beta \in \Gamma_i(\alpha)$. (It is easy to check that a G -distance-transitive graph is also distance-regular.) Γ is *distance-transitive* if it is G -distance-transitive for some group G . Further facts about distance-regular and distance-transitive graphs may be found in [1]. If $G \leq \text{Aut}(\Gamma)$, the full automorphism group of Γ , then we set $G^* = \langle G_\alpha : \alpha \in VT \rangle$. If Γ is G -distance-transitive, then either (i) Γ is bipartite and $|G : G^*| = 2$, or (ii) $G^* = G$. Smith [10] has shown that if for a G -distance-transitive graph Γ the permutation group (G, VT) is imprimitive, then a block system Ψ^G is of one of two possible types:

- (i) Ψ^G is the bipartition of the bipartite graph Γ , or
- (ii) $\Psi^G = \{\{\alpha\} \cup \Gamma_d(\alpha) : \alpha \in VT\}$ and Γ is antipodal.

In general a distance-regular graph Γ is called *antipodal* if the collection of sets $\{\{\alpha\} \cup \Gamma_d(\alpha) : \alpha \in VT\}$ forms a partition of VT , which is then called the *antipodal block system* of Γ . With any antipodal graph Γ we can associate a natural *quotient* graph, which we denote throughout by Γ' , whose vertex-set VT' is the set of antipodal blocks of Γ , two such being adjacent in Γ' whenever they contain adjacent vertices of Γ . (This is one of the cases in which one can imitate the permutation theoretic device of considering the quotient action on a set of maximal blocks, without losing the underlying graphical structure.) If Γ is antipodal with quotient graph Γ' , then Γ is called a $(1 + k_d)$ -fold *antipodal covering graph* of Γ' . (Since k_d is defined independently of $\alpha \in VT$, each antipodal block has the same size $r = (1 + k_d)$.) In [4] it was proved that r cannot exceed the valency

$k = k_1$ of Γ . Moreover, it was shown that the maximal case $r = k$ occurs only when either

(i) $k = 3$ and Γ is a three fold covering of Tutte's eight-cage, so $|V\Gamma| = 3.30$ or

(ii) $\Gamma' \cong K_{k,k}$ is the complete bipartite graph of valency k . We characterize here the k -fold antipodal covering graphs Γ of $K_{k,k}$. We prove the following theorem.

THEOREM A. *k -fold antipodal covering graphs of $K_{k,k}$ are in one-to-one correspondence with (nonisomorphic) triples $(\mathbf{P}, l(\infty), P(\infty))$, where \mathbf{P} is a projective plane of order k and $(l(\infty), P(\infty))$ is a distinguished flag of \mathbf{P} .*

THEOREM B. *Let Γ be a distance-transitive k -fold antipodal covering graph of $K_{k,k}$. Then the corresponding projective plane \mathbf{P} of order k is a self dual translation plane, with both $\mathbf{P} - l(\infty)$ and its dual being affine planes of rank three. \mathbf{P} may be coordinatised by a semifield.*

In particular the distance-transitive case arises only for k a prime power. We shall see that in the case of theorem B we can say much more, but we have been unable to deduce in general that \mathbf{P} is desarguesian, though the conditions obtained are such that we know of no counterexample to this natural conjecture. In particular cases one may conclude that the covering graph Γ is unique.

We use the notation of [3] for projective planes. In particular for a point P , and line l , of a projective plane \mathbf{P} , we write (P) for the set of all lines through P , and (l) for the set of all points on l ; if Q is another point, and m another line, of \mathbf{P} , then $P.Q$ denotes the join of P and Q , and $l.m$ denotes the meet of l and m . The Levi graph Γ of an incidence structure $\mathbf{I} = (\mathbf{B}, \mathbf{L}, I)$ has vertex-set $V\Gamma = \mathbf{B} \cup \mathbf{L}$, with adjacency in Γ defined by incidence in \mathbf{I} .

2. THE PROOF OF THEOREM A

The graph $\Gamma' \cong K_{k,k}$ has intersection array $(K_{k,k}) = \{k, k-1; 1, k\}$. From [4, Proposition 5.9] we know that a distance-regular k -fold antipodal covering graph Γ of $K_{k,k}$ has diameter 4 and intersection array

$$i(\Gamma) = \{k, k-1, k-1, 1; 1, 1, k-1, k\}.$$

Moreover, any graph with this intersection array is a k -fold antipodal covering graph of $K_{k,k}$.

LEMMA 2.1. *Let \mathbf{P} be a projective plane of order k and let Γ be the Levi graph of the incidence structure obtained by omitting a flag $(l(\infty), P(\infty))$, together with all points of $(l(\infty))$ and all lines of $(P(\infty))$. Then Γ is a distance-regular k -fold antipodal covering of $K_{k,k}$.*

The combinatorial regularity of the plane ensures that Γ is connected and distance-regular, and it is easy to see that Γ has the required intersection array. The antipodal blocks are of the form

$$\Psi_P = (P) - l(\infty), \text{ for each point } P \in (l(\infty)) - P(\infty),$$

$$\Psi_l = (l) - P(\infty), \text{ for each line } l \in (P(\infty)) - l(\infty).$$

Theorem A follows from the following Lemma.

LEMMA 2.2. *With any distance-regular k -fold antipodal covering graph Γ of $K_{k,k}$ we can associate a projective plane \mathbf{P} , with distinguished flag $(l(\infty), P(\infty))$.*

Proof. Denote by \mathbf{B} and \mathbf{L} the two sets of vertices in the bipartition of Γ , each having size k^2 . To the set \mathbf{B} we adjoin the "ideal" element $P(\infty)$, which we may take to be the object \mathbf{B} , together with the collection $P(1), P(2), \dots, P(k)$ of antipodal blocks contained in the set \mathbf{L} . Thus, we obtain the set \mathbf{B}^* of points. To the set \mathbf{L} we adjoin the "ideal" element $l(\infty)$, which we may take to be the object \mathbf{L} , together with the collection $l(1), l(2), \dots, l(k)$ of antipodal blocks contained in the set \mathbf{B} . Thus, we obtain the set \mathbf{L}^* of lines. Define incidence between elements of \mathbf{B}^* and \mathbf{L}^* as follows:

(i) incidence between elements of \mathbf{B} and elements of \mathbf{L} is given by adjacency in Γ ;

(ii) $P(\infty) Il(\infty)$;

(iii) incidence between other elements is given by set theoretic inclusion/containment; thus (a) $P(i) Il(\infty)$ and $P(\infty) Il(i)$, $1 \leq i \leq k$,

(b) if $P \in \mathbf{B}$ and $l \in \mathbf{L}$, then $PlI(i)$ whenever $P \in l(i)$, and $lIP(i)$ whenever $l \in P(i)$, $i = \infty, 1, \dots, k$.

From the properties of Γ it is then easy to check that $\mathbf{P} = (\mathbf{B}^*, \mathbf{L}^*, I)$ is a projective plane. ■

3. DISTANCE-TRANSITIVE GRAPHS AND RANK THREE PLANES

Throughout this section Γ denotes a G -distance-transitive graph which is a K -fold antipodal covering graph of $K_{k,k}$. \mathbf{P} is the projective plane associated with Γ as described in Lemma 2.2, and $(l(\infty), P(\infty))$ is the

distinguished flag of \mathbf{P} . \mathbf{A} is the affine plane $\mathbf{P} - l(\infty)$. For convenience we identify the relevant points and lines of \mathbf{P} with vertices of Γ . The group G induces a transitive group of permutations on the collection Ψ^G of antipodal blocks. Let K be the kernel of the action induced by G on the set of blocks. Since G acts distance-transitively on Γ , G induces a rank 5 permutation group on the set $V\Gamma = \mathbf{B} \cup \mathbf{L}$ of affine points and affine lines not through $P(\infty)$. Though major theorems are available [7–9] we can proceed from first principles; however, since proofs are essentially those of [5] (where a more general situation is studied) we merely state the relevant lemmas.

LEMMA 3.1. (i) G^* acts as a rank 3 group on the set of affine points, and on the set of affine lines not incident with $P(\infty)$.

(ii) The stabiliser in G of a line $l(i) \in (P(\infty))$ acts doubly transitively on $(l(i)) - P(\infty)$, $i = \infty, 1, \dots, k$.

(iii) The stabiliser in G of a point $P(i) \in (l(\infty))$ acts doubly transitively on $(P(i)) - l(\infty)$, $i = \infty, 1, \dots, k$.

(iv) For $1 \leq i, j \leq k$, $G_{l(i)P(j)}$ acts doubly transitively on

$$(l(i)) - P(\infty) \text{ and on } (P(j)) - l(\infty),$$

and transitively on $(l(\infty)) - \{P(\infty), P(j)\}$ and on $(P(\infty)) - \{l(\infty), l(i)\}$.

(v) Any element of $G - G^*$ induces a correlation of \mathbf{P} interchanging $P(\infty)$ and $l(\infty)$.

LEMMA 3.2. If K is nontrivial, then K is an elementary abelian p -group, for some prime p , which acts regularly on $(l(i)) - P(\infty)$, $1 \leq i \leq k$. K contains all translations of \mathbf{A} in the direction $P(\infty)$.

Let C be the kernel of the action of G on $(P(\infty))$ and D the kernel of the action of G on $(l(\infty))$. Then $C^g = D$, for each $g \in G - G^*$.

LEMMA 3.3. C is nontrivial if and only if D is nontrivial, in which case K is also nontrivial.

Using the double transitivity of the various actions in Lemma 3.1 one can generate all possible translations of \mathbf{A} .

LEMMA 3.4. D contains the full group D^* of translations of \mathbf{A} . Dually C contains the full group C^* of shears with axis $l \in (P(\infty))$. $C^* \cap D^* = C \cap D = K$ has order $k = p^r$, for some prime p .

Thus \mathbf{P} is $(l(\infty), l(\infty))$ -transitive and also $(P(\infty), P(\infty))$ -transitive, and

so may be coordinatised by a semifield \mathbf{F} [3, (3.1.22)(f)]. We briefly tie up what we have done with the study of rank 3 affine planes. Kallaher [8] has defined a *D-plane* to be a finite affine plane \mathbf{A}^* possessing a collineation group G^* with the following properties:

- (i) G^* is a rank 3 permutation group on the affine points;
- (ii) if \mathbf{P}^* is the extension of \mathbf{A}^* to a projective plane in the usual way, then G^* fixes some point $P(\infty)$ of the line at infinity $l(\infty)$ and acts transitively on $(l(\infty)) - P(\infty)$.

Kallaher proved that a *D-plane* of nonsquare order may be coordinatized by a semifield, and that with some difficulty. The extra strength of our hypotheses will be clear from the following lemma. For a *D-plane* \mathbf{A}^* let Γ^* be the graph of \mathbf{P}^* defined as in Lemma 2.1. With this notation we have the following.

LEMMA 3.5. *G^* has two orbits on $V\Gamma^*$. If P is an affine point, then $(G^*)_P$ acts transitively on $(\Gamma^*)_i(P) = \Gamma_i^*(P)$, $0 \leq i \leq 4$.*

Proof. Let P be an affine point. $(G^*)_P$ acts transitively on $(P) - PP(\infty) = \Gamma_1^*(P)$, whence $(G^*)_P$ acts transitively on the set $(l(\infty)) - P(\infty)$. Furthermore, $(G^*)_P$ acts transitively on the set $\Gamma_2^*(P)$ of affine points not lying on $P.P(\infty)$ by assumption, and also on the set $\Gamma_4^*(P)$ of points of $(P.P(\infty)) - \{P, P(\infty)\}$. Since $(G^*)_P$ acts transitively on $(l(\infty)) - P(\infty)$, Lemma 3.1 (iv) implies that $(G^*)_P$ acts transitively on the set $\Gamma_3^*(P)$ of affine lines not contained in $(P) \cup (P(\infty))$. Since \mathbf{P}^* may be assumed to have order at least 3, this last observation ensures that G^* acts transitively on the affine lines not contained in $(P(\infty))$. ■

Thus, our situation is essentially that of *D-planes* which admit a correlation stabilising the flag $(l(\infty), P(\infty))$. If we assume only that G^* is rank 3 on affine points, then for each $P(i) \in (l(\infty))$, $(G^*)_{P(i)}$ acts doubly transitively on the affine lines of $(P(i))$. If we further assume that G^* acts doubly transitively on $(l(\infty)) - P(\infty)$, then G^* is also rank 3 on the affine lines not contained in $(P(\infty))$, whence \mathbf{P} is $(P(\infty), P(\infty))$ -transitive and $(l(\infty), l(\infty))$ -transitive. (The plane of order 25 of M. Walker shows that this dual condition does not hold in general.)

$D^* = G(l(\infty), l(\infty))$ and $C^* = G(P(\infty), P(\infty))$ are elementary abelian, and for each i , $1 \leq i \leq k$, we have the direct decompositions $D^* = K \times D^*(P(i))$, and $C^* = K \times C^*(l(i))$. Since K is a minimal normal subgroup of G^* and since $C^* \cdot D^*$ is not abelian, we have $K = Z(C^*D^*) = \Phi(C^*D^*) = (C^*D^*)'$. By [3, (4.2.12)], \mathbf{P} is desarguesian whenever C^*D^* is the union of its abelian subgroups of order k^2 , in which case C^*D^* is isomorphic to the Sylow p -subgroup of $GL(3, p^r)$. Let

$C^*(l(i)) = \{c_1, c_2, \dots, c_k\}$, $D^*(P(j)) = \{d_1, d_2, \dots, d_k\}$, with $c_1 = d_1 = 1$. Then since $K = Z(C^*D^*)$, we have

$$[c_2 \cdot d_2, c_3 \cdot d_3] = [c_2, d_3][d_2, c_3].$$

Thus, c_2d_2 and c_3d_3 commute precisely when $[c_2, d_3] = [c_3, d_2]$. We have the following lemma.

LEMMA 3.6. *If for any $i = 1, 2, \dots, k$, $c \in C^*(l(i)) - 1$ is fixed, then $\{[c, d] : d \in D^*(P(j))\} = K$ for each $j = 1, 2, \dots, k$, and dually.*

Proof. One checks easily that $[c, d]$ fixes the point $P(j)$, but does not fix it linewise unless $d = 1$. Hence if $d \neq 1$, then $1 \neq [c, d] \in K$, whence both sets have the same cardinality and so are identical. ■

We may reorder the elements of $D^*(P(j)) - \{1, d_2\}$ in the following way: Given the pair c_2, d_2 we obtain for each c_m , $3 \leq m \leq k$, a unique element, d_m say, such that $[c_2, d_m] = [c_m, d_2]$. Thus \mathbf{P} is desarguesian whenever we may conclude that $[c_m, d_n] = [c_n, d_m]$, $2 \leq m, n \leq k$. The following lemma follows easily from what we have seen already.

LEMMA 3.7. (i) *If $l \in (P(i)) - l(\infty)$, $1 \leq i \leq k$, then*

$$G_l = D^*(P(i)) G_{P, P(i)}$$

for each $P \in (l) - P(i)$, and dually.

(ii) *If $1 \leq i, j \leq k$ and $P \in (l(j)) - P(\infty)$, then $G_{P, P(i)}$ acts transitively on each of $(l(\infty)) - \{P(i), P(\infty)\}$, $(l(j)) - \{P, P(\infty)\}$, $(P.P(i)) - \{P, P(i)\}$; $(P(\infty)) - \{l(\infty), l(j)\}$, $(P(i)) - \{l(\infty), P.P(i)\}$, $(P) - \{l(j), P.P(i)\}$.*

Lemma 3.7 (ii) describes the symmetrical conditions which the autotopism group A of \mathbf{F} must satisfy; in particular A has order divisible by $(k - 1)$ (where k is the order of \mathbf{F}), a property apparently not possessed by any known semifield.

It is possible, even in the desarguesian case, that G^* has order precisely $k^3(k - 1)$; there appear to be no theorems available which deduce from the existence of so small a collineation group, that \mathbf{P} is necessarily desarguesian. In special cases one can deduce that G is soluble, whence the various doubly transitive representations of G^* (see Lemma 3.1) are controlled by a theorem of Huppert [6]; (since a semifield of order $k = p^2$ is necessarily a field, the only exception arising in Huppert's theorem which concerns us is $k = 3^4$). for example, Burmester and Hughes [2] have proved that, under fairly general hypotheses, the autotopism group A of \mathbf{F} is soluble, whence $G^* = (C^*D^*)$. ($A \cap G^*$) is also soluble.

ACKNOWLEDGMENT

I should like to thank Norman Biggs for his encouragement, M. Burmester and M. Kallaher for helpful conversations, and the Science Research Council for their financial support.

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